第三讲：弹塑性理论基础

损伤力学基础研究生课程

任晓丹

同济大学建筑工程系

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本节主要内容

参考书

- Juan C. Simo and Thomas R.J. Hughes. 1997, Computational Inelasticity. Springer, New York.
- Jacob Lubliner. Plastic Theory. 2008, Dover Publication, New York.
- W.F. Chen and D.J. Han. 1988. Plasticity for structural Engineers. Springer-Verlag, New York.
- 陈惠发（余天庆、王勋文、刘再华译）, 2001. 土木工程材料的本构方程（卷 I、卷 II）. 华中科技大学出版社：武汉.

实验现象：屈服和强化

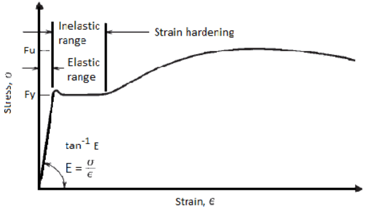


Figure 1

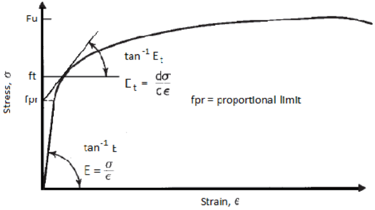
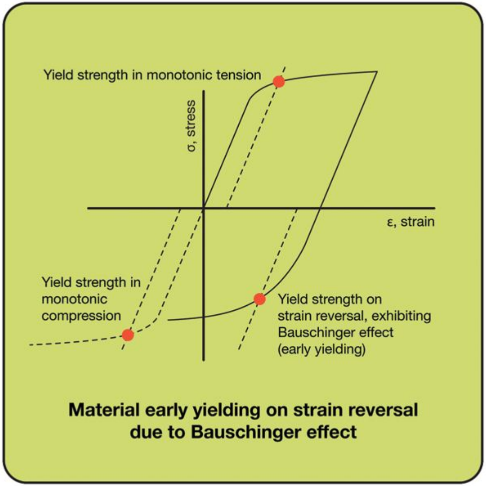


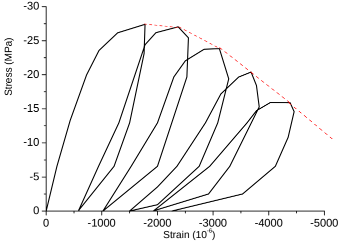
Figure 2

有明显屈服点的钢材和没有明显屈服点的钢材

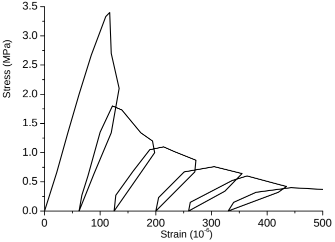
实验现象：包兴格效应


Yield strength in monotonic tension
σ, stress
ε, strain
Yield strength in
monotonic
compression
Yield strength on
strain reversal, exhibiting
Bauschinger effect
(early yielding)
Material early yielding on strain reversal
due to Bauschinger effect

实验现象：混凝土的塑性



重复加载受压实验[1]

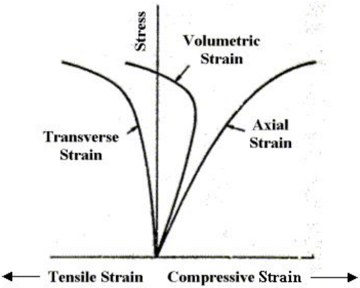


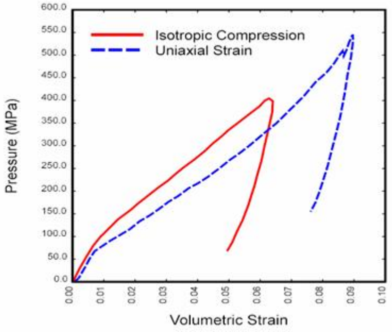
重复加载受拉实验[2]

[1] Karsan I D, Jirsa J O. Behavior of Concrete under Compressive Loadings. Journal of the Structural Division, 1969, 95(12):2543–2563.

[2] Taylor R L. FEAP: A finite element analysis program for engineering workstation Rep. No. UCB/SEMM-92 (Draft Version).

实验现象：混凝土的塑性

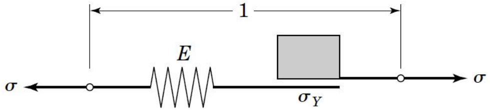




单轴受压实验　　约束受压实验

(http://www.fhwa.dot.gov/publications/research/infrastructure /pavements/05062/chapt2a.cfm)

一维弹塑性物理模型



弹簧摩擦元模型

- 应变弹塑性分解

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

- 应力应变关系

$$\sigma = E(\varepsilon - \varepsilon^p), \ \dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$$

- 任务：引入ε^p的表达式，建立以 σ、ε 及、ε^p 为状态量的闭合的理论体系

一维弹塑性模型数学分析

- 基本方程全量形式

$$\sigma = E(\varepsilon - \varepsilon^p)$$

- 补充方程全量形式

$$f(\sigma) = 0 \text{ or } f(\sigma, \varepsilon^p) = 0$$

方程数等于未知量数，不必要继续补充方程！

- 率形式

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) \quad , \quad \dot{f}(\sigma) = 0 \text{ or } \dot{f}(\sigma, \varepsilon^p) = 0$$

非线性硬化模型

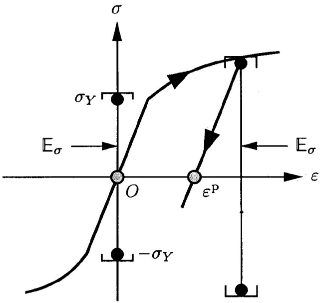
- 屈服条件

$$f(\sigma) = |\sigma| - q$$

- 硬化法则

$$q = q(\alpha), \quad \dot{q} = \frac{\partial q}{\partial \alpha}\dot{\alpha} := H\dot{\alpha}$$

- 塑性变量 α（同前）



非线性硬化模型

- 塑性演化：$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma} = \dot{\lambda}\,\mathrm{sign}(\sigma),\ \ \dot{\alpha} = |\dot{\varepsilon}^p| = \dot{\lambda}$
- 一致条件

$$f(\sigma) = 0$$

$$\Rightarrow \dot{f}(\sigma) = \frac{\partial f}{\partial \sigma}\dot{\sigma} - H\dot{\alpha} = \frac{\partial f}{\partial \sigma}E(\dot{\varepsilon} - \dot{\lambda}\frac{\partial f}{\partial \sigma}) - H\dot{\lambda} = 0$$

$$\Rightarrow \begin{cases} \dot{\lambda} = \frac{E}{E+H}\dot{\varepsilon}\ \mathrm{sign}(\sigma) \\ \dot{\varepsilon}^p = \frac{E}{E+H}\dot{\varepsilon} \end{cases}$$

- 增量本构关系

$$\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) = E^{ep}\dot{\varepsilon}, \quad E^{ep} = \begin{cases} E & \dot{\lambda} = 0 \\ \frac{EH}{E+H} & \dot{\lambda} > 0 \end{cases}$$

基本公式

- 应变弹塑性分解

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$$

- 应力应变关系

$$\boldsymbol{\sigma} = \mathbb{E}_0 : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$$

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}_0 : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p)$$

- 任务：
  - 引入 $\boldsymbol{\varepsilon}^p$ 的表达式，建立关于状态量 $\boldsymbol{\sigma}$、$\boldsymbol{\varepsilon}$ 及、$\boldsymbol{\varepsilon}^p$ 的闭合理论体系
  - 求解 $\dot{\boldsymbol{\varepsilon}}^p$、$\mathbb{E}^{ep}$，进而可进行结构非线性分析

弹塑性模型数学分析

- 多维弹塑性基本方程

$$\boldsymbol{\sigma} = \mathbb{E}_0 : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \text{ or } \dot{\boldsymbol{\sigma}} = \mathbb{E}_0 : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p)$$

- 补充方程（屈服条件）

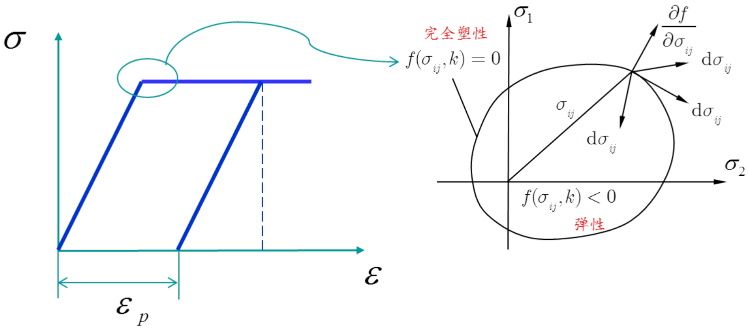
$$f(\boldsymbol{\sigma}) = 0 \text{ or } f(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = 0$$

方程数少于未知量数，须继续补充方程！

- 多维塑性演化

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\boldsymbol{\Gamma} = \dot{\lambda}\frac{\partial \tilde{f}}{\partial \boldsymbol{\sigma}} \rightarrow \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}$$

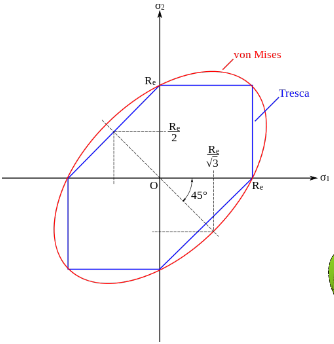
屈服函数

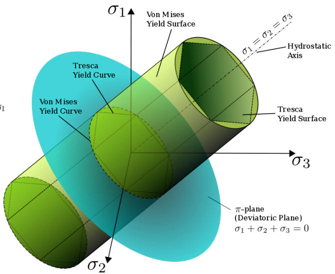


从一维空间的一个点到多维空间的曲面

屈服函数

- Tresca 屈服函数：
  $f = \frac{1}{2}\max_{i\neq j}(|\sigma_i - \sigma_j|) - q = 0$
- von Mises 屈服函数：$f = \sqrt{J_2} - q = 0$



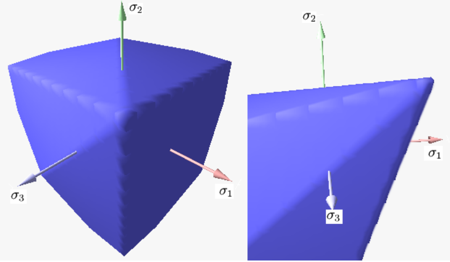


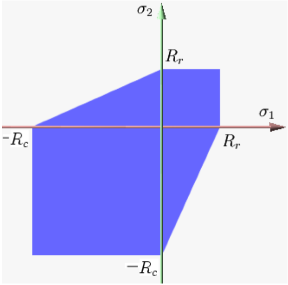
屈服函数

- Mohr-Coulomb 屈服函数：

$$f = \frac{m+1}{2}\max_{i\neq j}(|\sigma_i - \sigma_j| + K(\sigma_i + \sigma_j)) - q_c = 0$$

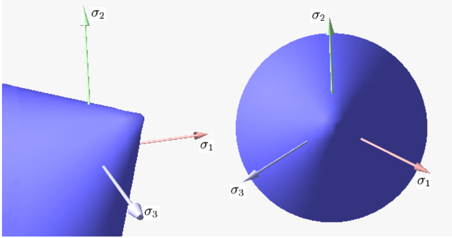
其中，$m = \frac{f_c}{f_t}$, $K = \frac{m-1}{m+1}$

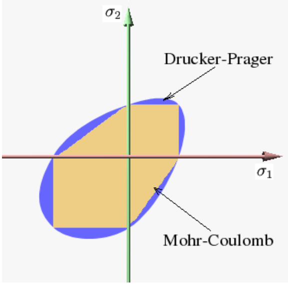




屈服函数

- Drucker–Prager 屈服函数：
  $f = \sqrt{J_2} - BI_1 - q = 0$



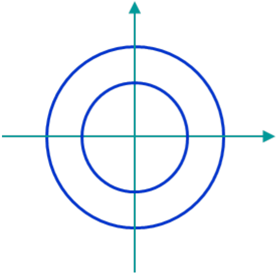


各向同性强化

考虑强化之后，屈服函数一般表达式为：$f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$，其中 $\boldsymbol{q} = (q_1, ..., q_N)$ 称为塑性内变量，是一系列塑性应变历史的标量函数。数学表达式为：$f(\boldsymbol{\sigma}, \boldsymbol{q}) = f(\boldsymbol{\sigma}) - q = f(\boldsymbol{\sigma}) - k(\boldsymbol{\alpha}) = 0$

其中 $k(\boldsymbol{\alpha})$ 为标量函数，那么不失一般性，可认为 $k(\boldsymbol{\alpha}) = k(\alpha)$。常用的塑性变量 α 的表达式有:

- Plastic work: $\int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^p$
- Equivalent plastic strain: $\int \sqrt{\frac{2}{3} d\boldsymbol{\varepsilon}^p : d\boldsymbol{\varepsilon}^p}$



运动强化

- 数学表达式为：

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = f(\boldsymbol{\sigma} - \overline{\boldsymbol{q}}) - k(\alpha) = 0$$

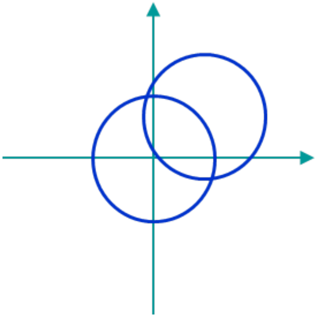
- 令 $\overline{\boldsymbol{q}} = \overline{k}(\alpha)$，$k(\alpha) = k$，即为经典随动强化
- Melan-Prager 模型

$$\overline{\boldsymbol{q}} = c\boldsymbol{\varepsilon}^p$$

- Ziegler 模型

$$\dot{\overline{\boldsymbol{q}}} = \dot{\mu}(\boldsymbol{\sigma} - \overline{\boldsymbol{q}})$$

- 记忆模型：$\dot{\overline{\boldsymbol{q}}} = c\boldsymbol{\varepsilon}^p - a\alpha\overline{\boldsymbol{q}}$



综合强化

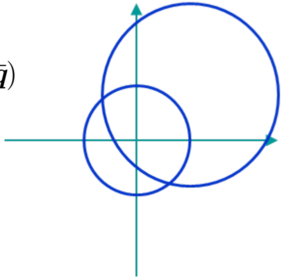
- 数学表达式为：

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = F(\boldsymbol{\sigma} - \overline{\boldsymbol{q}}; \boldsymbol{\alpha}) - k(\alpha) = 0$$

- Baltov-Sawczuk 模型

$$F(\boldsymbol{\sigma} - \overline{\boldsymbol{q}}; \alpha) = \frac{1}{2}(\boldsymbol{\sigma} - \overline{\boldsymbol{q}}) : \boldsymbol{A}(\boldsymbol{\alpha}) : (\boldsymbol{\sigma} - \overline{\boldsymbol{q}})$$

$$A_{ijkl} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl} + A\varepsilon^p_{ij}\varepsilon^p_{kl}$$



流动法则

- 考虑塑性演化可以表示为如下形式：

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\boldsymbol{r}(\boldsymbol{\sigma},\boldsymbol{q}) \\ \dot{\boldsymbol{q}} = -\dot{\lambda}\boldsymbol{h}(\boldsymbol{\sigma},\boldsymbol{q}) \end{cases}$$

- 加、卸载条件

$$\begin{cases} f < 0 \Rightarrow \dot{\lambda} = 0 \Rightarrow \text{elastic} \\ f = 0 \Rightarrow \begin{cases} \dot{f} < 0 \text{ and } \dot{\lambda} = 0 \Rightarrow \text{elastic unloading} \\ \dot{f} = 0 \text{ and } \dot{\lambda} = 0 \Rightarrow \text{neutral loading} \\ \dot{f} = 0 \text{ and } \dot{\lambda} > 0 \Rightarrow \text{plastic loading} \end{cases} \end{cases}$$

- Kuhn-Tucker 条件

$$f \le 0, \ \dot{\lambda} \ge 0, \ \dot{\lambda}\dot{f} = 0$$

一致条件

- 塑性一致性条件

$$f = 0 \Rightarrow \dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \dot{\boldsymbol{q}} = 0$$

$$\Rightarrow \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \left[\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}) \right] - \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$$

$$\Rightarrow \dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \dot{\boldsymbol{\varepsilon}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}}$$

- 塑性应变演化：$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}) = \left(\frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \boldsymbol{r}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}} \right) \dot{\boldsymbol{\varepsilon}}$

切线刚度

- 应力应变微分关系

$$\dot{\boldsymbol{\sigma}} = \mathbb{E}_0 : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p) = \left[\mathbb{E}_0 - \frac{\left(\mathbb{E}_0 : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) \otimes (\mathbb{E}_0 : \boldsymbol{r})}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}} \right] : \dot{\boldsymbol{\varepsilon}}$$

- 切线刚度张量

$$\mathbb{E}^{ep} = \begin{cases} \mathbb{E}_0 & \dot{\lambda} = 0 \\ \mathbb{E}_0 - \frac{\left(\mathbb{E}_0 : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) \otimes (\mathbb{E}_0 : \boldsymbol{r})}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{E}_0 : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}} & \dot{\lambda} > 0 \end{cases}$$

最大塑性功原理

- 最大塑性功原理：实际塑性变形使得下述塑性功取最大值

$$W = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p$$

- 约束条件： $f(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0$
- 采用拉格朗日乘子法构造复合函数

$$\Pi = W - \dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{q}) = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{q})$$

对复合函数求偏导

$$\frac{\partial \Pi}{\partial \boldsymbol{\sigma}} = \dot{\boldsymbol{\varepsilon}}^p - \dot{\lambda} \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{\sigma}} = 0 \Rightarrow \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{\sigma}}$$

- Kuhn-Tucker 条件：$f \leq 0, \ \ \dot{\lambda} \geq 0, \ \ \dot{\lambda} \dot{f} = 0$

最大塑性功原理

- 最大塑性功原理：实际塑性变形使得广义塑性功取最大值

$$W = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{q} \cdot \dot{\boldsymbol{\alpha}}$$

- 采用拉格朗日乘子法构造复合函数

$$\Pi = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{q} \cdot \dot{\boldsymbol{\alpha}} - \dot{\lambda} f(\boldsymbol{\sigma}, \boldsymbol{q})$$

- 求偏导得

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\alpha}} = \dot{\lambda} \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{q}} \Rightarrow \dot{\boldsymbol{q}} = \dot{\lambda} \left[\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}} \cdot \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{q}} \right]$$

- Kuhn-Tucker 条件

$$f \le 0, \quad \dot{\lambda} \ge 0, \quad \dot{\lambda} f = 0$$

相关塑性流动

- 正交流动：$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f(\boldsymbol{\sigma},\boldsymbol{q})}{\partial \boldsymbol{\sigma}}$, $\dot{\boldsymbol{q}} = \dot{\lambda}\left[\boldsymbol{H}\cdot\frac{\partial f(\boldsymbol{\sigma},\boldsymbol{q})}{\partial \boldsymbol{q}}\right]$, $\boldsymbol{H} = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}}$
- 塑性流动

$$\begin{cases} \dot{\lambda} = \dfrac{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbb{E}_0:\dot{\boldsymbol{\varepsilon}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial f}{\partial \boldsymbol{\sigma}}+\frac{\partial f}{\partial \boldsymbol{q}}\cdot\boldsymbol{H}\cdot\frac{\partial f}{\partial \boldsymbol{q}}} \\ \dot{\boldsymbol{\varepsilon}}^p = \left(\dfrac{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial f}{\partial \boldsymbol{\sigma}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial f}{\partial \boldsymbol{\sigma}}+\frac{\partial f}{\partial \boldsymbol{q}}\cdot\boldsymbol{H}\cdot\frac{\partial f}{\partial \boldsymbol{q}}}\right)\dot{\boldsymbol{\varepsilon}} \end{cases}$$

- 切线刚度：

$$\mathbb{E}^{ep} = \begin{cases} \mathbb{E}_0 & \dot{\lambda} = 0 \\ \mathbb{E}_0 - \dfrac{\left(\mathbb{E}_0:\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)\otimes\left(\mathbb{E}_0:\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)}{\frac{\partial f}{\partial \boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial f}{\partial \boldsymbol{\sigma}}+\frac{\partial f}{\partial \boldsymbol{q}}\cdot\boldsymbol{H}\cdot\frac{\partial f}{\partial \boldsymbol{q}}} & \dot{\lambda} > 0 \end{cases}$$

此时切线刚度张量满足主对称性

非相关塑性流动

- 非正交流动：$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\dfrac{\partial\tilde{f}(\boldsymbol{\sigma},\boldsymbol{q})}{\partial\boldsymbol{\sigma}}, \quad \dot{\boldsymbol{q}} = \dot{\lambda}\left[\boldsymbol{H}\cdot\dfrac{\partial\tilde{f}(\boldsymbol{\sigma},\boldsymbol{q})}{\partial\boldsymbol{q}}\right], \quad \boldsymbol{H} = \dfrac{\partial\boldsymbol{q}}{\partial\boldsymbol{\alpha}}$
- 塑性流动

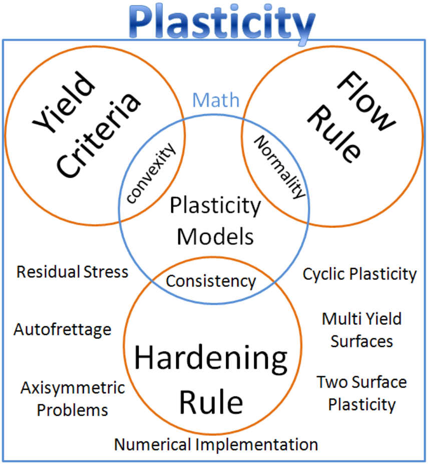
$$\begin{cases} \dot{\lambda} = \dfrac{\frac{\partial f}{\partial\boldsymbol{\sigma}}:\mathbb{E}_0:\dot{\boldsymbol{\varepsilon}}}{\frac{\partial f}{\partial\boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial\tilde{f}}{\partial\boldsymbol{\sigma}}+\frac{\partial f}{\partial\boldsymbol{q}}\cdot\boldsymbol{H}\cdot\frac{\partial\tilde{f}}{\partial\boldsymbol{q}}} \\ \dot{\boldsymbol{\varepsilon}}^p = \left(\dfrac{\frac{\partial f}{\partial\boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial\tilde{f}}{\partial\boldsymbol{\sigma}}}{\frac{\partial f}{\partial\boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial\tilde{f}}{\partial\boldsymbol{\sigma}}+\frac{\partial f}{\partial\boldsymbol{q}}\cdot\boldsymbol{H}\cdot\frac{\partial\tilde{f}}{\partial\boldsymbol{q}}}\right)\dot{\boldsymbol{\varepsilon}} \end{cases}$$

- 切线刚度

$$\mathbb{E}^{ep} = \begin{cases} \mathbb{E}_0 & \dot{\lambda} = 0 \\ \mathbb{E}_0 - \dfrac{\left(\mathbb{E}_0:\frac{\partial f}{\partial\boldsymbol{\sigma}}\right)\otimes\left(\mathbb{E}_0:\frac{\partial\tilde{f}}{\partial\boldsymbol{\sigma}}\right)}{\frac{\partial f}{\partial\boldsymbol{\sigma}}:\mathbb{E}_0:\frac{\partial\tilde{f}}{\partial\boldsymbol{\sigma}}+\frac{\partial f}{\partial\boldsymbol{q}}\cdot\boldsymbol{H}\cdot\frac{\partial\tilde{f}}{\partial\boldsymbol{q}}} & \dot{\lambda} > 0 \end{cases}$$

此时切线刚度张量不再满足主对称性。

塑性理论总结


Plasticity
Yield Criteria
Flow Rule
Math
convexity
Normality
Plasticity Models
Consistency
Hardening Rule
Residual Stress
Autofrettage
Axisymmetric Problems
Cyclic Plasticity
Multi Yield Surfaces
Two Surface Plasticity
Numerical Implementation

作业题

已知：

- 材料屈服函数为 $f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$;
- 四阶弹性刚度张量为 $\mathbb{E}_0$;
- 相关流动法则

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{\sigma}}, \quad \dot{\boldsymbol{q}} = \dot{\lambda}\left[\boldsymbol{H} \cdot \frac{\partial f(\boldsymbol{\sigma}, \boldsymbol{q})}{\partial \boldsymbol{q}}\right], \quad \boldsymbol{H} = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{\alpha}}$$

求 $\dot{\lambda}$，$\dot{\boldsymbol{\varepsilon}}^p$ 和 $\mathbb{E}^{ep}$ 的表达式。
注意：表达式须用应力、应变、屈服函数以及塑性变量等已知函数或符号表示，不得包含待求未知量；须给出推导过程，只有最后结果者不算完成作业；推导过程需严格遵循张量的运算规则

结束

- 邮箱名：damage_mechanics@163.com
- 密码：sunshanglixue